Contour Integration Cheat Sheet

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Definition 1 (Residue) *Assume that f has an isolated singularity at z_0. The residue of f at z_0, denoted* $\mathrm{Res}[f, z_0]$ *is the coefficient a_{-1} of $(z - z_0)^{-1}$ in the Laurent expansion of f at z_0:*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, 0 < |z - z_0| < \rho$$

Theorem 1 (The Residue Theorem) *Let D be a bounded set with a piecewise smooth boundary oriented in the counterclockwise direction. Assume that f is analytic on* $\mathrm{int}\, D$ *except at singularities $z_1, z_2, \ldots z_m \in \mathrm{int}\, D$. Then*

$$\int_{\partial D} f(z) dz = 2\pi i \sum_{i=1}^{m} \mathrm{Res}[f(z), z_j]$$

Strategies for finding residues 2-4 on this list straight from in 7.1 of *Complex Analysis* by Gamelin.

1. Partial Fractions decomposition
2. If $f(z)$ has a simple pole at z_0, then
$$\mathrm{Res}[f(z), z_0] = \lim_{z \to z_0} (z - z_0) f(z)$$
3. If f(z) has a double pole at z_0, then
$$\mathrm{Res}[f(z), z_0] = \lim_{z \to z_0} \frac{d}{dz} ((z - z_0)^2 f(z))$$
4. If f, g are analytic and g has a simple zero at z_0, then
$$\mathrm{Res}[\frac{f(z)}{g(z)}, z_0] = \frac{f(z_0)}{g'(z_0)}$$

Exercise 1 *Prove 2-4 above.*

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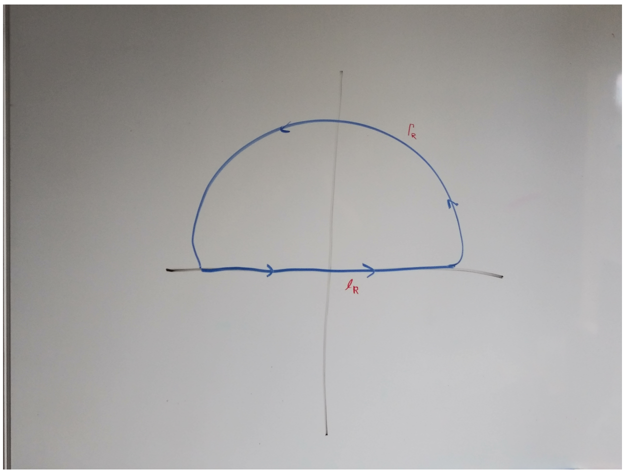

Semicircle Contour
limits: $R \to \infty$
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- integral from $-\infty$ to ∞
- $P(x)/Q(x)$ for Q, P polynomials with $\deg Q \geq \deg P + 2$
- $P(x)/Q(x)R(\sin(x), \cos(x))$ for P, Q, R polynomials with $\deg Q \geq \deg P + 1$
  - If in the previous bullet $\deg Q = \deg P + 1$, use Jordan's Lemma:

Lemma 1 (Jordan's Lemma) *Let Γ_R be the semicircle of radius R in the upper half plane. Then*

$$\int_{\Gamma_R} |e^{iz}||dz| < \pi$$
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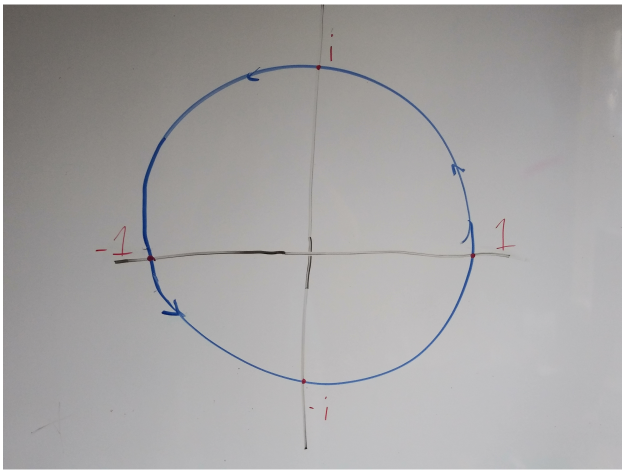

Unit Circle Contour
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- integral from 0 to 2π
- for rational functions of $\sin\theta, \cos\theta$
- Goal: turn into an integral of rational functions around unit circle. Use the substitutions $\sin(z) = (e^{iz} - e^{-iz})/2i, \cos(z) = (e^{iz} + e^{-iz})/2$
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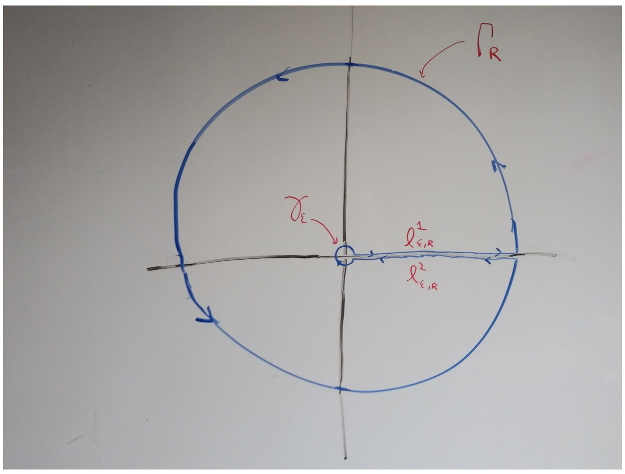

Keyhole Contour
limits: $\epsilon \to 0, R \to \infty$, integrating along $\ell^1_{\epsilon,R}, \ell^2_{\epsilon,R}$ is implicitly a limit as well
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- integral from 0 to infinity, denominator a polynomial in $P(x)$
- for integrand with a branch cut, typically z^a or $log(z)$
- the integrand will assume different values on either side of the branch
- In order for the integrand to have the right form on either side of the branch, irreducible factors of P should be $(x-a)$ for real a. One may first need to perform a change of variable if this is not the case
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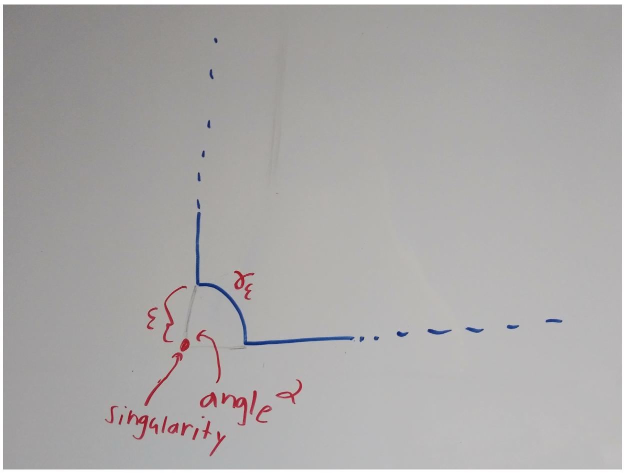

small arc angle α, also called a “contour indented at z_0”
limits: $\epsilon \to 0$
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- When integrating through a singularity on the real line and that singularity is a simple pole

Theorem 2 (Fractional Residue Theorem) *Let z_0 be a simple pole of f and let C_ϵ be the arc of angle α (in radians) of radius ϵ. Then*

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z)dz = \alpha i \operatorname{Res}[f(z), z_0)$$
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References: This cheat-sheet summarizes 7.1-7.5 and 7.7 of *Complex Analysis* by Gamelin. All theorem statements are from this text as well. Look in the textbook for proofs, worked examples, and more information.