Singularities and Extending Analytic Functions

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1 Singularities

1.1 Isolated Singularities

Assume that f has an *isolated* singularity at z_0. Then this singularity can be classified as one of three types:

1. Removable singularity: $\lim_{z\to z_0} f(z)$ exists
2. Pole: $\lim_{z\to z_0}(z-z_0)^m f(z)$ exists for some integer m
3. Essential: $\lim_{z\to\infty}(z-z_0)^m f(z)$ does not exist for any m

Exercise 1 • *What can you say about the Laurent series f at z_0 for each type of singularity?*

- *What kind singularities are the following at 0?*
  1. $e^{\frac{1}{z}}$ *(essential)*
  2. $\frac{\sin z}{z}$ *(removable)*
  3. $\frac{1}{\sqrt{z}}$ *(not isolated)*
  4. $\frac{1}{\sin z}$ *(pole)*
  5. $\log(z)$ *(not isolated)*

Theorem 1 (Riemann Removable singularity Theorem) *If an analytic function on the punctured disk around z_0 and is bounded on this neighborhood then z_0 is a removable singularity.*

Theorem 2 *Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole of $f(z)$ of order N iff $f(z) = g(z)/(z-z_0)^N$ where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.*

1.2 Branch Cuts

- Points in a branch cut are not isolated singularities, so this dichotomy does not apply
- Residue theory only works for isolated singularities–hence the indent on the keyhole contour!
- Keyhole contours still work with the fractional residue theorem

1.3 Theorems about Essential singularities

Theorem 3 (Casorati-Weierstrass Theorem) *Assume that f has an essential singularity at z_0 and let S be a punctured neighborhood of z_0. Then $f(S)$ is dense in $\mathbb{C}$.*

Theorem 4 (Picard's Big Theorem) *Assume that f has an essential singularity at z_0 and let S be a punctured neighborhood of z_0. Then f assumes every complex value infinitely many times on S with at most one exception.*

Theorem 5 (Picard's Little Theorem) *An entire non-constant function assumes every complex value with at most one exception.*

Note: The "at most one exception" is necessary, for instance consider $f(z) = e^z$.

- Using Picard's Little Theorem & Picard's Big Theorem on an exam is a bit heavy-handed, so avoid it if possible. However, using one knowing these theorems can help you find the answer, and writing an answer using one of these theorems is much better than no answer at all!
- Common pattern: to apply Picard's little theorem to an entire function, apply it to $f(z) = 1/z$

example: (1995 Jan # 3) Show that $e^z + 2$ has infinitely many zeroes

example: (2007 Jan #4) What is the most general entire function that takes each complex value once and only once in $\mathbb{C}$?

2 Extending analytic functions

Three useful theorems for this are below:

Theorem 6 (Liouville's Theorem) *A bounded entire function is constant.*

Theorem 7 (The identity principle) *Let g, f be analytic on an open set S and assume that f, g are equal on a set in S with an accumulation point. Then $f = g$.*

Theorem 8 (The Schwartz Reflection Principle (Gamelin)) *Let D be a be a domain that is symmetric with respect to the real axis, and let $D^+ = D \cap \{\operatorname{Im} z > 0\}$, $D^- = D \cap \{\operatorname{Im} z < 0\}$. Let $f(z)$ be an analytic function on D^+ such that $\operatorname{Im} f(z) \to 0$ as $z \in D^+$ tends to $D \cap \mathbb{R}$. Then $f(z)$ extends to be analytic on D, and the extension to D^- is given by*

$$f(z) = \overline{f(\overline{z})} z \in D^-$$

Example: `https://math.nyu.edu/student_resources/wwiki/index.php?title=Complex_Variables:_2011_January:_Problem_5`