Problems

1. (Gamelin) Suppose that $f(z)$ is an entire function such that $f(z)/z^n$ is bounded for $|z| > R$. Show that $f(z)$ is a polynomial of degree at most n. What can be said if $f(z)/z^n$ is bounded on the entire complex plane?

2. (1996 Sept. #5) If $f(z)$ is an entire function which assumes the values 0 and 1, show that for any complex number a and any real number $\epsilon > 0$ there is a point z_0 such that $|f(z_0) - a| < \epsilon$.

3. (Gamelin) Show that if u is a harmonic function on $\mathbb{R}$ that is bounded above, then u is constant.

4. (2021 January #4)(a) For $a, b \in \mathbb{C}$ linearly independent over $\mathbb{C}$, define the lattice $\Lambda = \{na + mb\colon (n,m) \in \mathbb{Z}^2\}$. A meromorphic function on $\mathbb{C}$ is elliptic (for the lattice Λ) if it satisfies $f(z) = f(z+\omega)$ for all $z \in \mathbb{C}, \omega \in \Lambda$. Show that an elliptic function which does not have poles is constant.

5. (Gamelin) Show that

$$\int_{-\infty}^{\infty} e^{-zt^2+2wt} dt = \sqrt{\frac{\pi}{z}} e^{\frac{w^2}{z}}, \quad z, w \in \mathbb{C}\, \mathrm{Re}(z) > 0$$

   where we take the principle branch of the square root. (Hint: you can use the fact that this integral equals 1 for $w = 0, z = 1$.)

6. (January 2021 #1)Find all holomorphic functions on $\mathbb{C}$ such that

$$f(1 + \frac{1}{n}) = \frac{1}{n}, \quad n \in \mathbb{N}$$

7. (1991 September # 6) Find all functions $f(z)$ satisfying
   - $f(z)$ is analytic on $\mathrm{Im}(z) > 0$
   - $f(z)$ is continous on $\{\mathrm{Im}\, z \geq 0\}$
   - $f(z)$ is real on the real axis
   - $|f(z)| > |\sin z|$ on $\mathrm{Im}\, z > 0\}$

8. (Fall 2020 #2) Find all entire functions f such that $|f(z)| \leq e^{xy}$ for all $z = x + iy \in \mathbb{C}$.

9. (1995 Septempter: P5) Find a function $f(z)$ that satisfies
   - $f(z)$ is analytic in the upper half plane, $\mathrm{Im}(z) > 0$,and continuous up to the real axis except at the origin
   - $f(z)$ is real when x is real and $x \neq 0$
   - $|f(z)| \leq \frac{C}{|x|^3}$ when $\mathrm{Im}(z) > 0$
   - $f(i) = 4i$

   Is this function unique? Why?

10. Show that the reflection in the circle $\{|z - z_0| = R\}$ is given by $z^* = z_0 + R^2(z - z_0)/|z - z_0|^2$.

11. Show that a reflection in a circle maps circles in the plane ot circles

12. What happens to angles between curves when they are reflected in an analytic arc?
13. Let $f(z)$ be an entire function whose modulus is constant on some circle. Show that $f(z) = c(z - z_0)^n$ for some $n \geq 0$ and some constant c, where z_0 is the center of the circle. (Hint: use problem 10)
14. Show that if $f(z)$ is meromorphic for $|z| < 1$ and $|f(z)| \to 1$ as $|z| \to 1$, then $f(z)$ is a rational function. (Hint: use problem 10)
15. Show that if v is a harmonic conjugate for u, then $-u$ is a harmonic conjugate fro v