Extending Analytic Functions

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1 Basic tools

Theorem 1 (Liouville's Theorem) *A bounded entire function is constant.*

Theorem 2 (The identity principle) *Let g, f be analytic on an open set S and assume that f, g are equal on a set in S with an accumulation point. Then $f = g$.*

Definition 1 *Assume that f is analytic on the punctured disk $0 < |z - z_0| < \epsilon$. Then we call z_0 a* removable singularity *if $\lim_{z\to z_0} f(z)$ exists.*

A function with a removable singularity at z_0 can be extended to an analytic function on $|z - z_0| < \epsilon$.

Theorem 3 (Riemann Removable singularity Theorem) *If an analytic function on the punctured disk around z_0 and is bounded on this neighborhood then z_0 is a removable singularity.*

2 Harmonic Functions

Theorem 4 *If $f(x + iy) = u(x, y) + iv(x, y)$ is an analytic function, then u, v are harmonic.*

Definition 2 *If u is harmonic and $u + iv$ is analytic, then v is called a* harmonic conjugate *of u*

The Cauchy-Riemann equations are a useful tool for finding harmonic conjugates

Exercise 1 *The harmonic conjugate of u is unique up to an additive constant*

Theorem 5 *Let $h(x, y)$ be a continuous function on a domain D. Then $h(x, y)$ is harmonic on D iff for every circle $C = \{(x, y) \colon |(x, y) - (x_0, y_0)| = r\}$ contained in D,*

$$u(x_0, y_0) = \frac{1}{2\pi r} \int_C h(x, y)$$

This property is called the mean value property.

3 The Schwartz Reflection Principle

3.1 Reflecting across the real line

Theorem 6 (Schwartz Reflection Principle) *Let D be an open set that is symmetric with respect to the real axis, and let $D^+ = D \cap \{Imz > 0\}$ be the part of D in toe open upper half-plane.*

- *If $u(z)$ is a real-valued harmonic function on D^+ for which $u(z) \to 0$ as $z \in D^+$ approaches any value of $D \cap \mathbb{R}$, then u can be extended to a harmonic function on D through*

$$u(\overline{z}) = -u(z)$$

- *If $f(z)$ is a complex-valued analytic function on D^+ for which* $\operatorname{Im}(f(z)) \to 0$ *as $z \in D^+$ approaches any value of $D \cap \mathbb{R}$, then f can be extended to an analytic function on D through*

$$f(\overline{z}) = \overline{f(z)}$$

Idea of proof: use mean value property/morera's theorem

3.2 Reflecting across a curve

- Assume that we have an analytic function f defined on a set with boundary curve γ and $f(z)$ approaches real values as z approaches γ
- if we can map γ to the real axis in an analytic fashiion, then we can reflect across γ
- if we can reflect across γ *locally*, we can still reflect across gamma
- Formally: γ is an *analytic curve* if every point of γ has an open neighborhood U for which there is a disk D centered on the real line and a conformal map $\zeta \colon D \to U$ for which $\zeta(D \cap \mathbb{R}) = U \cap \gamma$.
- Conjugation in D induces a map from U to itself, $z(\zeta)^* = z(\overline{\zeta})$
- It can be shown that the * operation is unique

Written exam example: `https://math.nyu.edu/student_resources/wwiki/index.php?title=Complex_Variables:_2011_January:_Problem_5`

4 The Poisson Kernel

Most important fact: Say we have a continuous $\mathbb{C}$-valued function h defined on $\partial\mathbb{D}$. Then we can extend it to all of

$$\tilde{h}(re^{i\theta}) = \int_{-\pi}^{\pi} h(e^{i\varphi}) P_r(\theta - \varphi) \frac{d\varphi}{2\pi}$$

This extension is harmonic. Any such extension is unique. (Can be shown by the maximum principle)

Basic properties:

a) Setting $z = re^{i\theta}$ for $r < 1$, $P_r(\theta) = \frac{1-|z|^2}{|1-z|^2} = \frac{1-r^2}{1+r^2-2r\cos\theta} = \operatorname{Re}\frac{1+z}{1-z}$

b) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta) d\theta = 1$

c) $P_r(\theta) > 0$, for $r < 1$

d) $P_r(-\theta) = P_r(\theta)$

e) $P_r(\theta)$ is increasing for $-\pi \le \theta \le 0$ and decreasing for $0 \le \theta \le \pi$

f) For fixed $\delta > 0$, $\max\{P_r(\theta) \colon \delta \le |\theta| \le \pi\} \to 0$ as $r \to 1$

Comments:

- a) $P_r(\theta)$ is harmonic in (r, θ)
- b),c) imply the poisson kernel is a probability density function
- d) symmetry
- f) implies that the mass concentrates as 0 as $r \to 1$